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О МЕРАХ БЛИЗОСТИ И РАЗЛИЧИМОСТИ СМЕШАННЫХ КВАНТОВЫХ СОСТОЯНИЙ

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ON MEASURES OF CLOSENESS AND DISTINGUISHABILITY OF MIXED QUANTUM STATES

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Обсуждаются различные меры близости и различимости матриц плотности. Детально анализируется синус угла между состояниями. Как показало исследование литературы по данному вопросу, эта мера близости еще не была подробно изучена. Тем не менее, она естественно возникает в контексте передачи и обработки информации на квантовых носителях. Синус угла между состояниями имеет много притягательных свойств, что делает его хорошей мерой для сравнения матриц плотности. Если синус угла между состояниями мал, то будут близкими распределения вероятностей, генерируемые этими состояниями для любого обобщенного измерения. Обсуждаются сходства и различия данной меры со следовой дистанцией и метрикой Бюрса. Степень совпадения – обобщение на случай смешанных квантовых состояний квадрата модуля скалярного произведения векторов состояния. В связи с этим показано, что существует бесконечно большое число аналогичных мер с теми же свойствами, за исключением мультипликативности.

Different measures of closeness and distinguishability of density matrices are considered. The sine distance between mixed quantum states is thoroughly investigated. This distance does not seem to have been previously analyzed in the literature. Nevertheless, the sine distance naturally arises in the context of quantum information processing. As it is shown, this distance has many attractive features that make it a good measure of closeness of density matrices. We prove that if the sine distance between two quantum states is small then probability distributions generated by these states for any POVM measurement are close to each other. Relations and distinctions of the sine distance with the trace distance and the Bures metric are discussed. The fidelity is well-known generalization of the square-overlap of pure states to the case of mixed states. In this connection, we show that there is infinite number of similar measures with the same properties except the multiplicativity.

Over the last few decades there have been impressive theoretical and experimental advances in quantum information processing. The quantum cryptography can already be regarded as realized new technology for data protection [1]. The quantum factoring [2] and quantum searching [3] are important advances in quantum computation. A design of efficient algorithm for task of quantum communication and computation demands that we compare results of probe quantum operations. So quantitative measures of closeness and distinguishability of quantum (generally mixed) states are needed. Even in the case of pure states we must consider more than one way to fit the problem of comparison of quantum states. In the present work the sine of angle between two states is considered as a distance measure. It will be referred as to “sine distance”. The use of the sine distance as distance measure provides new and fruitful viewpoint on the state-dependent cloning [4]. More recently, the authors of [5] found that the above distance naturally arises in the context of quantum computation. But the sine distance has been not studied previously as independent notion. After brief outline of background material, we describe the basic properties of the sine distance. Further, the sine distance is considered in the context of quantum operations. In the present work by the quantum states we mean that ones are all normalized. As every, we define the angle $\delta(x,y)\in[0;\pi/2]$ between the pure states $|x\rangle$ and $|y\rangle$ by the natural relation $\delta(x,y)\equiv\arccos\,|\langle x|y\rangle|$. Pure states $|x\rangle$ and $|y\rangle$ are identical if and only if $\delta(x,y)=0$.

Definition 1. Sine distance between two pure states is defined by

$$C(x,y)\equiv\sin\delta(x,y)\,. \tag{1}$$

But all the real devices are inevitably exposed to noise. So the pure states used will evolve into mixed states. How close are two mixed quantum states? We known that the square-overlap $|\langle x|y\rangle|^2$ is the probability that $|y\rangle$ passes the yes/no test of “being the state $|x\rangle$”. However, there is no evident analog of yes/no test for mixed states. Nevertheless, we can extend to mixed states a few notions which are useful in the case of pure states. This is provided by the concept of purifications. If the quantum system S is considered then we append system Q, which is a copy of S. Further, we can imagine that a mixed state ρ of S arises by partial trace operation from pure state of extended system S+Q. Namely, there is a pure state $|X\rangle$, called “purification”, for which [6]

$$\rho=Trace_Q\{\,|X\rangle\langle X|\,\}\,. \tag{2}$$

For any mixed states its purification can be made, and for given one such a pure state is not unique [6]. The angle $\Delta(\rho,\sigma)\in[0;\pi/2]$ between mixed states ρ and σ is then defined by

$$\Delta(\rho,\sigma)\equiv\min\ \delta(X,Y)\,, \tag{3}$$

where the minimization is over all purifications $|X\rangle$ of ρ and $|Y\rangle$ of σ [7]. The properties of angle between mixed states are listed in [7]. We are now able to extend the notion of “sine distance” to the case of mixed states.

Definition 2. Sine distance between mixed states ρ and σ is defined by

$$C(\rho,\sigma)\equiv\sin\Delta(\rho,\sigma)\,. \tag{4}$$

The sine distance obeys the following properties [8]. It ranges between 0 and 1, and $C(\rho, \sigma)$=0 if and only if $\rho=\sigma$. It is symmetric, i.e. $C(\rho, \sigma)=C(\rho, \sigma)$. It satisfies the triangle inequality, i.e. $C(\rho, \sigma)= C(\rho, \omega)+ C(\rho, \omega)$. So the sine distance is a metric on the space of quantum states It turns out that for any quantum operation the quantity $C(\rho, \sigma)$ estimates the difference between probabilities of processes beginning with inputs ρ and σ. Namely, the following statement takes place [8].

Theorem 1. If the set $\{E_\mu\}$ of operators specifies a quantum operation $\rho \to \$(\rho)$ then

$$\left| \; Trace\{\$(\rho)\} - Trace\{\$(\sigma)\} \;\right| \le \; C(\rho,\sigma)\,, \tag{5}$$

$$\sum\nolimits_\mu \left| \; Trace\{E_\mu \;\; \rho \;\; E_\mu\} - Trace\{E_\mu \;\; \sigma \;\; E_\mu\} \;\right| \le \; 2 \;\; C(\rho,\sigma)\,. \tag{6}$$

The measurement is an important type of quantum operation. In this case the input and output spaces are the same. As pointed out by Everett [9], a general treatment of all observations by the method of projection operators is impossible. The most general quantum measurement is called a "positive operator valued measure", or POVM. A POVM with M distinct outcomes is specified by a set of M positive operators A_μ satisfying

$$\sum\nolimits_\mu A_\mu = I\,, \tag{7}$$

where I denotes the identity operator. Note that the number M of different outcomes is not limited above by the dimensionality of Hilbert space, in contrast to von Neumann measurements. If the system S is prepared in stateρ, then the probability of μ 'th outcome is

$$p_\mu(\rho) \equiv Trace\{\rho \quad A_\mu\}\,. \tag{8}$$

With each POVM element A_μ one can associate an ideal quantum operation $\$_\mu$ defined by

$$\$_\mu(\rho) \equiv \sqrt{A_\mu} \quad \rho \quad \sqrt{A_\mu}\,. \tag{9}$$

Applying now the statement of Theorem 1 we obtain the following result.

Theorem 2. For arbitrary POVM measurement and any two states ρ and σ there holds

$$\left| \; p_\mu(\rho) - p_\mu(\sigma) \;\right| \le C(\rho,\sigma)\,, \tag{10}$$

$$\sum\nolimits_\mu \left| \; p_\mu(\rho) - p_\mu(\sigma) \;\right| \le 2 \quad C(\rho,\sigma)\,. \tag{11}$$

Thus, if the sine distance $C(\rho, \sigma)$ is small then probability distributions generated by states ρ and σ for arbitrary (generalized) measurement are close to each other. So the sine distance can be regarded as a good measure of closeness of quantum states.

The sine distance is closely related to the trace distance [10]. Moreover, in the case of pure states these two distance measures are equal to each other. Recall the definition of the trace distance. Let $|A|$ denote the positive square root of $A^\dagger A$ (for any positive operator exists a unique positive square root [11]). Then the trace distance $D(\rho, \sigma)$ between states ρ and σ is defined by

$$D(\rho,\sigma) \equiv \frac{1}{2} \quad Trace \;\; \left| \; \rho - \sigma \; \right|. \tag{12}$$

The trace distance has many attractive properties that makes it a suitable measure of closeness of quantum states. The sine distance can simply be expressed in the terms of fidelity function. Recall that the fidelity function generalizes the square-overlap. More precisely, for given states ρ and σ of system S the fidelity is defined as

$$F(\rho,\sigma) \equiv \max \;\; \left| \; \langle X | Y \rangle \; \right|^2, \tag{13}$$

where the maximum is taken over all purifications $|X\rangle$ of ρ and $|Y\rangle$ of σ [6]. Using Eqs. (3), (4) and (13), it is easy to check that

$$C(\rho,\sigma) = \sqrt{1 - F(\rho,\sigma)}\,. \tag{14}$$

As it is mentioned above, the square-overlap is the natural and physically motivated measure of distinguishability of pure states. We wish to extend the notion of square-overlap to the case of mixed states. It is natural to impose the following axioms: (I) $0 \le G(\rho,\sigma) \le 1$ and G(ρ, σ)=0 if and only if σ=ρ; (II) $G(\rho, \sigma)= G(\sigma, \rho)$; (III) If $\rho = |x\rangle \; \langle x|$ and $\sigma = |y\rangle \; \langle y|$ then $G(\rho,\sigma) = \left| \; \langle x|y \rangle \; \right|^2$.

The fidelity function is the well-known measure satisfying the axioms (I)–(III). However, there is an infinite number of those measures that obey these axioms. Indeed, let p and q be positive real numbers meeting $p+q=1$. We define a measure

$$G(\rho,\sigma) = p \quad \sqrt{\;1 - C^2(\rho,\sigma)} + q \quad \sqrt{\;1 - D^2(\rho,\sigma)}\,, \tag{15}$$

where $C(\rho, \sigma)$ is the sine distance and $D(\rho, \sigma)$ is the trace distance. Both the $C(\rho, \sigma)$ and $D(\rho, \sigma)$ ranges between 0 and 1. Further, $C(\rho, \sigma)=0$ as well as $D(\rho, \sigma)=0$ if and only if ρ=σ. So the measure defined by Eq. (15) satisfies the axiom (I). Both the sine distance and the trace distance are symmetric, so that the defined by Eq. (15) function is also symmetric. In the case of pure states $\rho = |x\rangle \; \langle x|$ and $\sigma = |y\rangle \; \langle y|$ we have

$$C(\rho,\sigma) = D(\rho,\sigma) = \sqrt{\;1 - \left| \; \langle x|y \rangle \; \right|^2}\,. \tag{16}$$

Due to Eq. (16) and $p+q=1$, the measure defined by Eq. (15) satisfies the axioms (I)-(III). It should be pointed out that the fidelity is multiplicative function. That is,

$$F(\rho \otimes \omega, \sigma \otimes \chi) = F(\rho,\sigma) \quad F(\omega,\chi)\,. \tag{17}$$

On the contrary, the measure defined by Eq. (15) is not multiplicative.

In addition, the sine distance is closely related to the Bures metric. As it is well known, this metric is equal to the square root of the quantity $2 \;\; (\; 1 - \sqrt{F} \;\;)$, where F denotes the fidelity [6]. In terms of the angle between quantum states the Bures metric $B(\rho, \sigma)$ is expressed as

$$B(\rho,\sigma) = 2 \quad \sin\left(\Delta(\rho,\sigma)/2\right). \tag{18}$$

Thus, for small values of angle $\Delta(\rho, \sigma)$ the Bures metric is approximately equal to the sine distance. Which of these distances is most preferable? One may scarcely maintain that such a formulation of question is justified. Rather, some distance should be preferred in a first kind of tasks, other distance should be preferred in a second kind of tasks, and so on. But the following must be emphasized. The sine distance lies in the interval [0;1], whereas the angle lies in $[0;\pi/2]$ and the Bures measure lies in $[0;\sqrt{2}]$. As the range of distance values, the interval [0;1] seems to be more natural. In addition, the sine distance between two states allows to estimate directly a distinction between their observable effects. So the sine distance is a reliable measure of closeness of quantum states.

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